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Paper

Optimal Force Distribution for the Legs of a Quadruped Robot*

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Abstract: The real-time force control of a quadruped robot involves optimization of an underdetermined force system subjected to both equality and inequality constraints. A new method for optimal force distribution for the legs of a quadruped robot is presented in this paper. It is characterized by transforming the friction constraints from the nonlinear inequalities into a combination of linear equalities and linear inequalities, by eliminating the linear equality constraints from the original problem to reduce the problem size, and by solving a quadratic optimization problem to meet the needs for quality of solution. The technique is compared with the existing QP (Quadratic Programming) Method and Analytical Method to show its superior performance in terms of the problem size, quality of solution and the scope of application. The effectiveness of the proposed method is illustrated by giving some simulation results of optimal foot force distribution for a quadruped robot.

Keywords: Quadruped Robot, Optimal Force Distribution, Quadratic Programming, Friction Constraints

1. Introduction

As a kind of legged vehicles, quadruped robots can be applied in the work space with rough terrain, e.g., map building on an uneven ground, hazardous tasks like landmine searching and removing, volcano data collection, etc. The fact that the control strategy of a quadruped robot must take the force distribution of the legs into account, is considered to be very important. A quadruped robot has not only a kinematic topology, but also a redundant actuation. This arises due to the general existance of three actuated joints in each leg, resulting in more controlled actuators than the degree-of-freedom of the body motion. Therefore, mathematically there exist fewer force moment balance equations than unknown design variables, and the solution to these equations is not unique. Moreover, such a robot has physical constraints that can only be represented mathematically as inequalities due to the nature of the contacts involved, i.e., normal contact forces between the feet and the ground cannot be negative, and the magnitude of the tangential force at each foot cannot exceed the maximum force of static friction. In addition, the torque of each joint must lie within the allowed range. Therefore, force distribution for legs involves optimization of force for the legs, considering the inequality constraints, same as that of cooperating manipulators, mechanical hands, etc.

In recent two decades, many researchers have studied this problem and have developed different algorithms for the optimal solutions [1]–[25], in which there are mainly four representative methods:

1. Linear-Programming (LP) Method [1]–[5], [25], in which this problem is formulated as a linear-programming problem, by replacing friction cone with a piecewise linear pyramid so as to express friction constraints by linear inequalities. But this method is difficult to be applied for complex systems in real-time and it leads easily to discontinuous solutions even under smooth changes in constraints [8];
2. Compact-Dual LP (CDLP) Method [6], this method results in a smaller problem size by using the compact-dual linear programming, but it cannot overcome the discontinuous problem [8];
3. Quadratic Programming (QP) Method [7], [8], it is superior to LP and CDLP in quality of solution and also can be implemented in real-time because the time for obtaining a solution does not depend on an initial guess. However, the existing numerical examples were limited to finger system without mentioning the applicability to walking robot. In a general walking robot, the number of constraints is larger than that in a finger system. Furthermore, the quadruped robot has troublesome kinematic topology. Therefore, it may be difficult to implement this method in real-time without reducing the problem size [15];
4. Analytical Method [9]–[13], this method is mainly applied to the walking robot. It is characterized by finding a relation among feet forces in order to prevent legs from slipping, and consequently adding equality constraints so that the underdetermined force system may be transformed into a determined system after combining inverse dynamic equations with the equality constraints. However, it attempts only to prevent leg slippage, neglecting the other inequality constraints.

In addition, some researchers proposed the optimal force distribution scheme of multiple cooperating robots by combining the Dual Method with the QP [14], [15].

For a quadruped robot, the optimization process involves three aspects to be considered: the first is to prevent legs from slipping; the second is to avoid the discontinuity of

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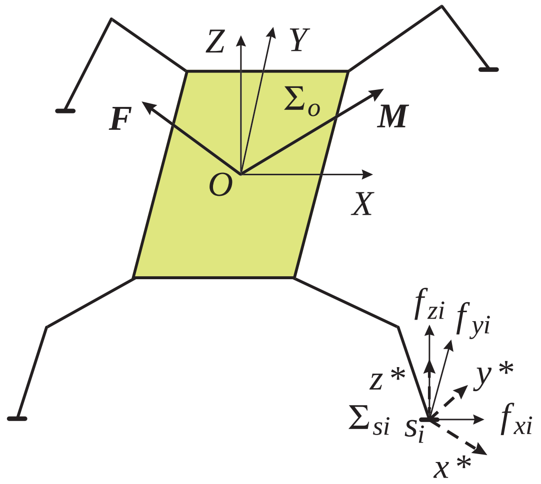

Fig. 1 The force system acting on a quadruped robot

the foot forces for such a kinematic topology system; and the third is to make the foot forces of swing leg increase smoothly from zero, after contacting the ground so as to avoid impact. Based on the existing methods, a new algorithm on optimal force distribution for legs of a quadruped robot is proposed in this paper. The main idea of our work is originated from the combination of the QP Method with the reduction of the problem size. The present approach can eliminate the disadvantages of the Analytical Method and the QP Method in optimization of force distribution for a quadruped robot.

The rest of this paper is organized as follows. In Section 2 the problem formulation is presented, which includes mathematical description of the problem. Section 3 gives reduction of the problem size, continuous solution for the problem, and optimal solution for the problem. The example and simulation results will be presented in Section 4.

2. Problem Formulation

The force system acting on a quadruped robot is shown in **Fig. 1**. For simplicity, only the force components of a foot are presented here. As has been usual in such work, rotational torques at the feet are neglected. Let Σ_o $(O-XYZ)$ be the coordinate frame fixed at the robot body in which the body is located in the X–Y plane and Σ_{si} $(s_i - x^*y^*z^*)$ denote the coordinate frame fixed at the foot i, in which the leg i lies in the x^*–z^* plane and z^*-axis is normal to the support surface of the foot which is assumed to be parallel to the X–Y plane. $\boldsymbol{F} = [F_x \ \ F_y \ \ F_z]^T \in \Re^3$ and $\boldsymbol{M} = [M_x \ \ M_y \ \ M_z]^T \in \Re^3$ respectively denote the robot body force vector and moment vector, which result from the gravity and the external force acting on the robot body. Define f_{xi}, f_{yi}, and f_{zi} as the components of the force acting on the supporting foot i in the directions of X, Y, and Z in Σ_o, respectively. The number of supporting feet, n, can vary between 3 and 4 for a quadruped robot. The force/moment quasi-static equilibrium equation of the robot can be written as

$$\boldsymbol{A}\boldsymbol{G} + \boldsymbol{W} = \boldsymbol{0} \tag{1}$$

with

$$\boldsymbol{G} = [f_{x1} \ \ f_{y1} \ \ f_{z1} \ \ \cdots f_{zn}]^T,$$

$$\boldsymbol{W} = [\boldsymbol{F}^T \ \ \boldsymbol{M}^T]^T,$$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{I}_3 & \cdots & \boldsymbol{I}_3 \\ \boldsymbol{B}_1 & \boldsymbol{B}_2 & \cdots & \boldsymbol{B}_n \end{bmatrix},$$

$$\boldsymbol{B}_i = \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix},$$

where $\boldsymbol{I}_3 \in \Re^{3\times3}$ is the identity matrix and $\boldsymbol{G} \in \Re^9$ or $\in \Re^{12}$ is the foot force vector, corresponding to three or four legs supporting, respectively. $\boldsymbol{A} \in \Re^{6\times9}$ or $\in \Re^{6\times12}$ is a coefficient matrix which is a function of the positions of the supporting feet, where $\boldsymbol{B}_i \in \Re^{3\times3}$ is a skew symmetric matrix consisting of (x_i, y_i, z_i), which is the position coordinate of the supporting foot i in Σ_o. $\boldsymbol{W} \in \Re^6$ is a total body force/moment vector.

It is clear that Eq. (1) is an underdetermined system and its solution is not unique. In other words, the feet forces have many solutions according to the equilibrium equation. However, the feet forces of the quadruped robot, in fact, must meet the needs for the following physical constraints; otherwise they would be invalid.

First, all supporting feet should not slip when the robot walks on the ground. It results in the following constraints:

$$\sqrt{f_{xi}^2 + f_{yi}^2} \le \mu f_{zi}, \quad (i = 1, \cdots, n), \tag{2}$$

where μ is the static coefficient of friction of the ground.

Second, since the feet forces are generated from the corresponding actuators of joints, it must take the physical limits of the joint torques into account. Then it follows that

$$-\boldsymbol{\tau}_{i\max} \le \boldsymbol{J}_i^T \boldsymbol{R}_i \begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{zi} \end{bmatrix} \le \boldsymbol{\tau}_{i\max}, \tag{3}$$

for $i = 1, \cdots, n$, where $\boldsymbol{J}_i \in \Re^{3\times3}$ is the Jacobian of the leg i, $\boldsymbol{\tau}_{i\max} \in \Re^3$ is the maximum joint torque vector of the leg i, and $\boldsymbol{R}_i \in \Re^{3\times3}$ is the orientation matrix of Σ_{si} with respect to Σ_o.

Finally, to assure definite foot contact with the ground, there must exist f_{zi} such that,

$$f_{zi} \ge 0, \quad (i = 1, \cdots, n). \tag{4}$$

Therefore, the problem of the foot force planning can be described as a nonlinear programming problem. Clearly, it is difficult to solve such a nonlinear programming problem for real-time foot force distribution of a quadruped robot with complex constraints.

3. Reduction of the Problem Size and the Solution

As shown in **Fig. 2**, most of researchers substituted the inscribed pyramid for the friction cone formulated by Eq. (2). Thus, the nonlinear friction constraints are approximately expressed by the linear inequalities

$$f_{xi} \le \mu' f_{zi}, \quad f_{yi} \le \mu' f_{zi}, \quad (i = 1, \cdots, n), \tag{5}$$

where $\mu' = \sqrt{2}\mu/2$ for the (conservative) inscribed pyramid. Therefore, the nonlinear programming can be transformed into a linear programming, e.g., see [2], [6], and [7].

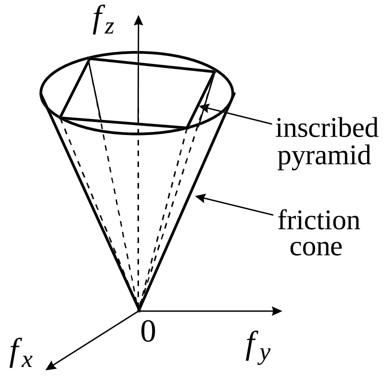

Fig. 2 The friction cone and pyramid

To minimize the possibility of slipping, the force distribution should be such that the maximum of the ratios of tangential to normal forces at the feet is minimized. It has been shown that, for a multi-legged robot, the force ratios are minimized if all ratios are equal to the global ratio [10]. It is on the basis of this idea that Liu and Wen [13] found the relationship among the feet forces and transformed the friction constraints from the nonlinear inequalities into a set of linear equalities so that the underdetermined force system may become a determined system by combining Eq. (1) with the set of linear equalities. However, it should be noted that they obtained such ratios under the assumption that all horizontal force components of the robot body force/moment are zero. In the strict sense of the word, it is only the optimal condition that prevents leg slippage for the case of no horizontal force. So its scope of application is limited. Moreover, the method only seeks for the minimum ratios of tangential to normal forces at the feet, at the cost of neglecting the constraints expressed by Eqs. (3) and (4). For example, consider that a quadruped robot, whose mechanism is described in **Figs. 3** and **4**, walks on the plain ground in a periodic crawl gait, with the stride length of each swing leg to be 200 [mm], and the robot body force/moment, $F_z = 200$ [N], $F_y = 20$ [N]. Using the Analytical Method presented in the reference [13], the results of the force distribution for the legs in a gait cycle f_{zi} $(i = 1, \cdots, 4)$ can be obtained as shown in **Fig. 5**. It is clear that in this case the constraints $f_{z3} \geq 0$ and $f_{z4} \geq 0$ are violated.

Here we synthesize the advantages of the existing methods, that is, part of components of the feet forces satisfy the relationship of global ratio of tangential to normal forces at the robot body, and let the other components satisfy linear inequality constraints as Eq. (5). For example, defining f_{xi} $(i = 1, \cdots, n)$ to be the former, and f_{yi} $(i = 1, \cdots, n)$ to be the latter, then we obtain

$$f_{xi} = k_{xz} f_{zi}, \quad (i = 1, \cdots, n), \tag{6}$$

$$f_{yi} \leq \mu^* f_{zi}, \quad (i = 1, \cdots, n), \tag{7}$$

where $k_{xz} = F_x/F_z$ is the global ratio of X- to Z-direction forces at the robot body. μ^* is the given coefficient for friction constraints. According to Eq. (2), we have

$$\mu^* = \sqrt{\mu^2 - k_{xz}^2}. \tag{8}$$

Similarly, making a permutation of f_{xi} and f_{yi}, then

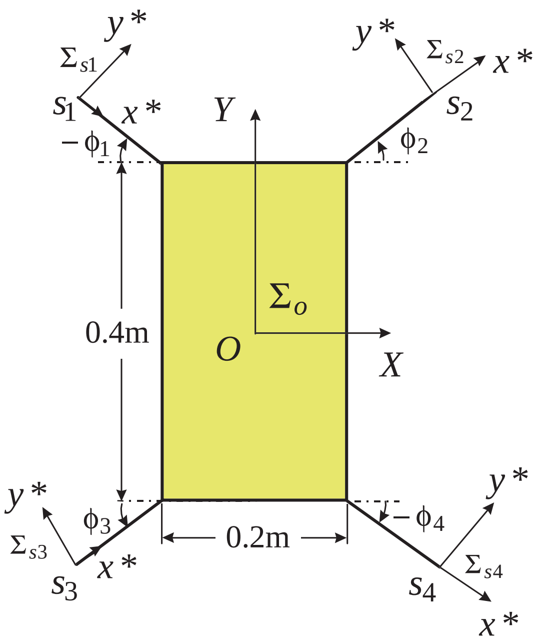

Fig. 3 The vertical view of a quadruped robot

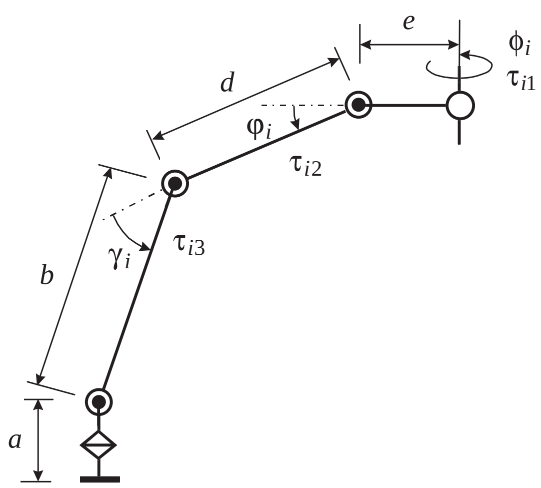

Fig. 4 The basic mechanism of a leg

Eqs. (6) and (7) can be rewritten as

$$f_{yi} = k_{yz} f_{zi}, \quad (i = 1, \cdots, n), \tag{9}$$

$$f_{xi} \leq \mu^* f_{zi}, \quad (i = 1, \cdots, n), \tag{10}$$

where $k_{yz} = F_y/F_z$ and $\mu^* = \sqrt{\mu^2 - k_{yz}^2}$.

Clearly, the force distribution becomes a linear problem and its computation is considerably reduced by replacing Eq. (2) with Eqs. (6) and (7) or Eqs. (9) and (10). In particular, it can overcome the demerit of the Analytical Method as mentioned above.

3.1 Continuous solution for the problem

A quadruped robot can realize its crawling using three and four legs alternatively to support its body. However, it must be taken into account how to avoid the discontinuity of the feet forces. The consideration should be twofold: the first is to assure that the foot force of the swing leg continuously transits while the leg moves from free to placement on the ground; the second is to make the foot force increase

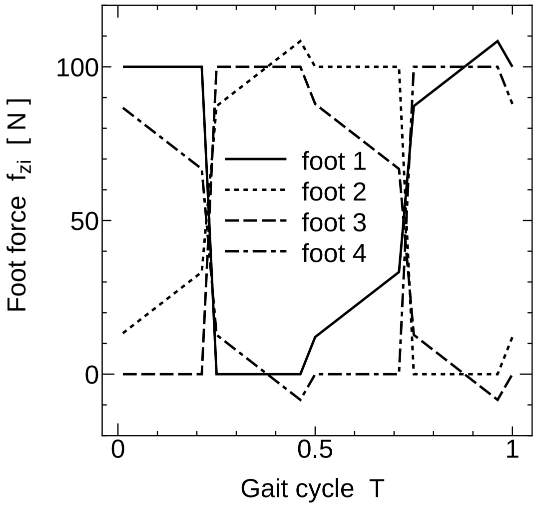

Fig. 5 The simulation result by using the analytical method

smoothly from zero in order to avoid impact resulting from the placement of the swing leg.

For the convenience of explanation, let (t_j^-, t_j^+) denote the time period from the time t_j^- at which the leg j (the swing leg) is lifted, to the time t_j^+ at which the leg j is placed; (t_j^+, t_k^-) be the time period from the time t_j^+ at which the leg j is placed, to the time t_k^- at which the leg k (next swing leg) is lifted. It is obvious that the robot uses three legs to support on the ground in the time period (t_j^-, t_j^+), whereas it uses four legs to do so during the time period (t_j^+, t_k^-).

In the time period $(t_j^-, t_j^+), n = 3$, so that $\boldsymbol{G}$ and $\boldsymbol{A}$ become a vector of 9×1 and a matrix of 6×9, respectively. Equation (1) contains nine unknown variables with six equations. Although there are nine equations by combining it with Eq. (6) or (9), actually there are only eight independent equations in existence,

$$\hat{\boldsymbol{A}}\hat{\boldsymbol{G}} + \hat{\boldsymbol{W}} = \boldsymbol{0}, \tag{11}$$

where $\hat{\boldsymbol{A}} \in \Re^{8\times9}$ is the resulting matrix of $\boldsymbol{A}$ after combination. $\hat{\boldsymbol{G}} \in \Re^9$ is the foot force vector. $\hat{\boldsymbol{W}} \in \Re^8$ is the resulting vector of $\boldsymbol{W}$ after combination.

Thus, the force distribution can be reduced to the force planning subjected to the equality constraints expressed by Eq. (11) and the inequality constraints expressed by Eqs. (3), (4) and (7) or (10).

In the time period $(t_j^+, t_k^-), n = 4$, so that $\boldsymbol{G}$ is a 12-dimensional vector, and $\boldsymbol{A}$ is a matrix of 6×12. For the sake of continuity of solution, the foot forces of the leg j denoted by $\boldsymbol{f}_j = [f_{xj} \;\; f_{yj} \;\; f_{zj}]^T \in \Re^3$ should be changed smoothly from $\boldsymbol{f}_j(t_j^+) = \boldsymbol{0}$ to $\boldsymbol{f}_j(t_k^-)$, which can be determined according to the force planning for three supporting legs at the time t_k^-. Therefore, the foot force of the leg j in the time period (t_j^+, t_k^-) can be expressed as

$$\boldsymbol{f}_j = \delta(t)\boldsymbol{f}_j(t_k^-), \tag{12}$$

where $\delta(t)$ is any desired continuous scalar function varying from $\delta(t_j^+) = 0$ to $\delta(t_k^-) = 1$. Thus, $\boldsymbol{f}_j$ is regarded to be known in this time period and Eq. (1) can be rewritten as

$$[\tilde{\boldsymbol{A}} \quad \boldsymbol{A}_j] \begin{bmatrix} \tilde{\boldsymbol{G}} \\ \boldsymbol{f}_j \end{bmatrix} + \boldsymbol{W} = \boldsymbol{0}, \tag{13}$$

where $\tilde{\boldsymbol{A}} \in \Re^{6\times9}$, $\tilde{\boldsymbol{G}} \in \Re^9$, and $\boldsymbol{A}_j \in \Re^{6\times3}$. It follows that

$$\tilde{\boldsymbol{A}} \;\; \tilde{\boldsymbol{G}} + (\boldsymbol{A}_j\boldsymbol{f}_j + \boldsymbol{W}) = \boldsymbol{0}. \tag{14}$$

Let $\tilde{\boldsymbol{W}} = \boldsymbol{A}_j\boldsymbol{f}_j + \boldsymbol{W} \in \Re^6$. We then have

$$\tilde{\boldsymbol{A}}\tilde{\boldsymbol{G}} + \tilde{\boldsymbol{W}} = \boldsymbol{0}. \tag{15}$$

Since there are only eight independent equations among Eqs. (15) and (6) or (9), we obtain the equality constraints of the force distribution in the form expressed in Eq. (11) by combining Eq. (15) with Eq. (6) or (9). Note here that $\hat{\boldsymbol{A}}$ is the resulting matrix of $\tilde{\boldsymbol{A}}$ after combination, and $\hat{\boldsymbol{W}}$ is the resulting vector of $\tilde{\boldsymbol{W}}$ after combination. As a result, the force distribution in the time period (t_j^+, t_k^-) can be converted into one for three legs except for the leg j, under the consideration of continuous solution for the problem.

3.2 Optimal solution for the problem

The solution to the inverse dynamic equations of a quadruped robot is not unique, but it can be chosen in an optimal manner by introducing an objective function. The approach taken here is to minimize the sum of the weighted torque efforts of the robot, which results in the following objective function [8], [15]:

$$f(\boldsymbol{G}) = \boldsymbol{p}^T\boldsymbol{G} + \frac{1}{2}\boldsymbol{G}^T\boldsymbol{Q}\boldsymbol{G}, \tag{16}$$

with

$$\boldsymbol{p}^T = \left[\hat{\boldsymbol{\tau}}_1^T\boldsymbol{J}_1^T \cdots \hat{\boldsymbol{\tau}}_n^T\boldsymbol{J}_n^T\right] \in \Re^{3n},$$

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{J}_1\boldsymbol{q}_1\boldsymbol{J}_1^T & \cdots & \boldsymbol{0} \\ \vdots & \ddots & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{J}_n\boldsymbol{q}_n\boldsymbol{J}_n^T \end{bmatrix} \in \Re^{3n\times3n},$$

where $\hat{\boldsymbol{\tau}}_i$ is the joint torque vector due to the weight and inertia of the leg i, $\boldsymbol{J}_i$ is the Jacobian of the leg i, and $\boldsymbol{q}_i$ is a positive definite diagonal weighting matrix of the leg i. This objective function is strictly convex.

Because the time for obtaining a solution does not depend on an initial guess, a quadratic programming is superior to linear programming in both speed and quality of the obtained solution [8]. The general linear-quadratic programming problem of the force distribution for legs of a quadruped robot is stated by

$$\text{minimize} \quad \boldsymbol{p}^T\hat{\boldsymbol{G}} + \frac{1}{2}\hat{\boldsymbol{G}}^T\boldsymbol{Q}\hat{\boldsymbol{G}}, \tag{17}$$

$$\text{subject to} \quad \hat{\boldsymbol{A}}\hat{\boldsymbol{G}} + \hat{\boldsymbol{W}} = \boldsymbol{0}, \tag{18}$$

$$\boldsymbol{B}\hat{\boldsymbol{G}} \le \boldsymbol{C}, \tag{19}$$

where $\hat{\boldsymbol{G}} \in \Re^9$ is a vector of the design variables (the feet forces) and Eq. (17) is the objective function, while Eqs. (18) and (19) represent the equality and inequality constraints, respectively. It should be pointed out that, Eq. (18) denotes Eq. (11), and Eq. (19) is the resulting inequality constraints for the combination of Eqs. (3), (4) and (7) or (10) where

$$\boldsymbol{B} = \left[\boldsymbol{B}_1^T \ \boldsymbol{B}_2^T \ \boldsymbol{B}_3^T \ \boldsymbol{B}_4^T\right]^T \in \Re^{9\times 24},$$

$$\boldsymbol{C} = \left[\boldsymbol{\tau}_{1\max}^T \ \boldsymbol{\tau}_{2\max}^T \ \boldsymbol{\tau}_{3\max}^T \ -\boldsymbol{\tau}_{1\max}^T \ -\boldsymbol{\tau}_{2\max}^T \ -\boldsymbol{\tau}_{3\max}^T \ 0\ 0\ 0\ 0\ 0\ 0\right]^T \in \Re^{24},$$

with

$$\boldsymbol{B}_1 = \begin{bmatrix} \boldsymbol{J}_1^T\boldsymbol{R}_1 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J}_2^T\boldsymbol{R}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{J}_3^T\boldsymbol{R}_3 \end{bmatrix} \in \Re^{9\times 9},$$

$$\boldsymbol{B}_2 = \begin{bmatrix} -\boldsymbol{J}_1^T\boldsymbol{R}_1 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{J}_2^T\boldsymbol{R}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{J}_3^T\boldsymbol{R}_3 \end{bmatrix} \in \Re^{9\times 9},$$

$$\boldsymbol{B}_3 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \in \Re^{3\times 9},$$

$$\boldsymbol{B}_4 = \begin{bmatrix} 0 & 1 & -\mu^* & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\mu^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\mu^* \end{bmatrix} \in \Re^{3\times 9}.$$

To simplify the problem, the general solution is obtained to eliminate the linear equality constraints from the problem, before quadratic-programming technique is applied. Because there are eight linear independent equations, Eq. (18) may be transformed into a desired row-reduced echelon form [6], which has an identity matrix $\boldsymbol{I}_8 \in \Re^{8\times 8}$ in the first eight columns of the resulting matrix such that

$$[\boldsymbol{I}_8 \quad \hat{\boldsymbol{A}}_r] \begin{bmatrix} \hat{\boldsymbol{G}}_b \\ \hat{\boldsymbol{G}}_r \end{bmatrix} + \hat{\boldsymbol{W}}_r = \boldsymbol{0}, \tag{20}$$

where $\hat{\boldsymbol{A}}_r \in \Re^8$ is the remaining column of the matrix $\hat{\boldsymbol{A}}$ after transformation. $\hat{\boldsymbol{G}}_b \in \Re^8$ is the partial vector of $\hat{\boldsymbol{G}}$. $\hat{\boldsymbol{G}}_r \in \Re$ is the unknown element of $\hat{\boldsymbol{G}}$ which denotes the design variable. $\hat{\boldsymbol{W}}_r \in \Re^8$ is the resulting vector of $\hat{\boldsymbol{W}}$ after transformation. Equation (20) may be rewritten by the following form

$$\boldsymbol{I}_8\hat{\boldsymbol{G}}_b + \hat{\boldsymbol{A}}_r\hat{\boldsymbol{G}}_r + \hat{\boldsymbol{W}}_r = \boldsymbol{0}, \tag{21}$$

which yields

$$\hat{\boldsymbol{G}}_b = -\hat{\boldsymbol{W}}_r - \hat{\boldsymbol{A}}_r\hat{\boldsymbol{G}}_r. \tag{22}$$

Finally, it results in

$$\hat{\boldsymbol{G}} = \begin{bmatrix} \hat{\boldsymbol{G}}_b \\ \hat{\boldsymbol{G}}_r \end{bmatrix} = \begin{bmatrix} -\hat{\boldsymbol{W}}_r \\ 0 \end{bmatrix} + \begin{bmatrix} -\hat{\boldsymbol{A}}_r \\ 1 \end{bmatrix} \hat{\boldsymbol{G}}_r. \tag{23}$$

Now let $\hat{\boldsymbol{G}}_0 = [-\hat{\boldsymbol{W}}_r^T \ \ 0]^T \in \Re^9$ and $\boldsymbol{N} = [-\hat{\boldsymbol{A}}_r^T \ \ 1]^T \in \Re^9$, then Eq. (23) becomes

$$\hat{\boldsymbol{G}} = \hat{\boldsymbol{G}}_0 + \boldsymbol{N}\hat{\boldsymbol{G}}_r. \tag{24}$$

Substituting Eq. (24) into Eqs. (17) and (19), the linear quadratic programming problem can be expressed by

$$\text{minimize} \quad f(\hat{\boldsymbol{G}}_r), \tag{25}$$

$$\text{subject to} \quad \boldsymbol{B}\boldsymbol{N}\hat{\boldsymbol{G}}_r \le \boldsymbol{C} - \boldsymbol{B}\hat{\boldsymbol{G}}_0, \tag{26}$$

where

$$f(\hat{\boldsymbol{G}}_r) = \boldsymbol{p}^T\hat{\boldsymbol{G}}_0 + \frac{1}{2}\hat{\boldsymbol{G}}_0^T\boldsymbol{Q}\hat{\boldsymbol{G}}_0 + \boldsymbol{p}^T\boldsymbol{N}\hat{\boldsymbol{G}}_r + \frac{1}{2}\hat{\boldsymbol{G}}_0^T\boldsymbol{Q}\boldsymbol{N}\hat{\boldsymbol{G}}_r + \frac{1}{2}\hat{\boldsymbol{G}}_r^T\boldsymbol{N}^T\boldsymbol{Q}\hat{\boldsymbol{G}}_0 + \frac{1}{2}\hat{\boldsymbol{G}}_r^T\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N}\hat{\boldsymbol{G}}_r.$$

Since $\hat{\boldsymbol{G}}_r$ is a single design variable denoted by x, the optimal force distribution can be further presented as

$$\text{minimize} \quad a_0x^2 + a_1x + a_2 \tag{27}$$

$$\text{subject to} \quad x \in [b_1\ b_2] \tag{28}$$

where

$$a_0 = \frac{1}{2}\boldsymbol{N}^T\boldsymbol{Q}\boldsymbol{N},$$

$$a_1 = \boldsymbol{p}^T\boldsymbol{N} + \frac{1}{2}\hat{\boldsymbol{G}}_0^T\boldsymbol{Q}\boldsymbol{N} + \frac{1}{2}\boldsymbol{N}^T\boldsymbol{Q}\hat{\boldsymbol{G}}_0,$$

$$a_2 = \boldsymbol{p}^T\hat{\boldsymbol{G}}_0 + \frac{1}{2}\hat{\boldsymbol{G}}_0^T\boldsymbol{Q}\hat{\boldsymbol{G}}_0.$$

in which $[b_1 \ \ b_2]$ denotes the bound resulted from Eq. (26). Since it is clear that $a_0 > 0$ because of the positive-definite matrix $\boldsymbol{Q}$, there must be an optimal solution for the force distribution.

4. Numerical Example

The basic mechanism, size and parameters of a quadruped robot are shown in Figs. 3 and 4, where $a = 0.043$ [m], $b = 0.2$ [m], $d = 0.155$ [m], and $e = 0.045$ [m]. There are three actuated joints ϕ_i, φ_i, and γ_i in the leg i, whose torques are denoted as τ_{i1}, τ_{i2}, and τ_{i3}, for $i = 1, \cdots, 4$, respectively. Assume the weight of leg is ignorable, then the Jacobian of the leg i can be expressed by

$$\boldsymbol{J}_i = \sigma\left[\boldsymbol{J}_{i1} \quad \boldsymbol{J}_{i2} \quad \boldsymbol{J}_{i3}\right], \tag{29}$$

for $i = 1, \cdots, 4$, where

$$\boldsymbol{J}_{i1} = \begin{bmatrix} 0 \\ -[a + bS(\varphi_i+\gamma_i) + dS\varphi_i] \\ -[a + bS(\varphi_i+\gamma_i)] \end{bmatrix},$$

$$\boldsymbol{J}_{i2} = \begin{bmatrix} e + dC\varphi_i + bC(\varphi_i+\gamma_i) \\ 0 \\ 0 \end{bmatrix},$$

$$\boldsymbol{J}_{i3} = \begin{bmatrix} 0 \\ bC(\varphi_i+\gamma_i) + dC\varphi_i \\ bC(\varphi_i+\gamma_i) \end{bmatrix},$$

and $\sigma = 1$ for $i = 1$ or 3, $\sigma = -1$ for $i = 2$ or 4, and $S*$ and $C*$ denote $\sin *$ and $\cos *$, respectively. From Fig. 3, the orientation matrix of Σ_{si} with respect to Σ_o can be obtained by

$$\boldsymbol{R}_i = \begin{bmatrix} \cos\phi_i & \sin\phi_i & 0 \\ -\sin\phi_i & \cos\phi_i & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{30}$$

The known crawl gait is a prerequisite for the optimization technique presented above because all joint positions are to be determined only from the known gait by using the inverse kinematics [26], [27]. Here we assume the quadruped robot is crawling in the direction of Y-axis of Σ_o on the uneven ground with the static coefficient of friction $\mu = 0.05$. The initial position coordinates of feet in Σ_o are (in meter): $(-0.3, 0.2, -0.243)$, $(0.3, 0.2, -0.243)$, $(-0.3, -0.2, -0.243)$, and $(0.3, -0.2, -0.243)$, respectively. The sequence of swing leg generated is $3 \to 1 \to 4 \to 2$. The stride length of every swing leg is 0.2 [m], the next Z-coordinates of the feet placements are $z_1 = -0.243$ [m], $z_2 = -0.238$ [m], $z_3 = -0.243$ [m], and $z_4 = -0.248$ [m], respectively. The duty factors (the time fraction of a gait cycle time T in which a leg is in the supporting phase [13]), $\beta_i = 0.8$, for $i = 1, \cdots, 4$, i.e., the period of (t_j^-, t_j^+) is $T/5$, and the period of (t_j^+, t_k^-) is $T/20$. Then we assume that the maximum joint torque vector (in Nm) is $\boldsymbol{\tau}_i = [40\ \ 40\ \ 40]^T$ and the body force/moment is: for case I, $F_x = 0$, $F_y = 0$, $F_z = -250$ [N], $\boldsymbol{M} = \boldsymbol{0}$; for case II, $F_x = -5$ [N], $F_y = 10$ [N], $F_z = -250$ [N], $m_x = 1$ [Nm], $m_y = m_z = 0$.

4.1 Simulation result for case I

For the case I, $k_{yz} = 0$, $\mu^* = 0.05$, $\delta(t)$ here is a linear scalar function. The objective function of the optimization problem taken here is to minimize the internal force [8], [15], i.e., $\boldsymbol{p} = \boldsymbol{0}$ (the zero vector), and $\boldsymbol{Q} = \boldsymbol{I}$ (the identity matrix). Solving Eqs. (27) and (28) and substituting the solution into Eq. (23), the feet forces can be obtained. **Figure 6** shows the foot force distribution in a gait cycle. Since $f_{xi} = f_{yi} = 0$ $(i = 1, \cdots, 4)$ the ratio of tangential to normal forces at the feet in a gait cycle is

$$\frac{\sqrt{f_{xi}^2 + f_{yi}^2}}{f_{zi}} = 0.$$

It is found that the feet of the robot don't slip during their standing phases because the ratio of tangential to normal forces at every foot is less than the static coefficient of friction. Furthermore, it is shown that the foot force of the swing leg increases smoothly from zero to the desired force so that its discontinuity is effectively excluded as shown in Fig. 6. Substituting the feet forces into the following equation, the joint torques of the leg can be determined

$$\begin{bmatrix} \tau_{i1} \\ \tau_{i2} \\ \tau_{i3} \end{bmatrix} = \boldsymbol{J}_i^T \boldsymbol{R}_i \begin{bmatrix} f_{xi} \\ f_{yi} \\ f_{zi} \end{bmatrix}, \quad i = 1, \cdots, 4. \tag{31}$$

Here we have shown the result of the leg 2 in **Fig. 7**.

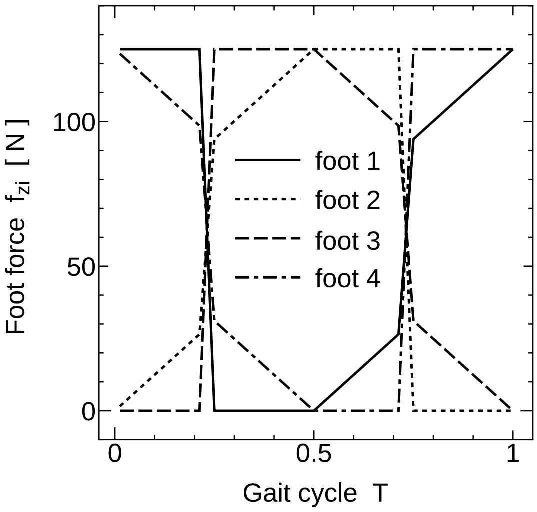

Fig. 6 The foot force distribution for case I

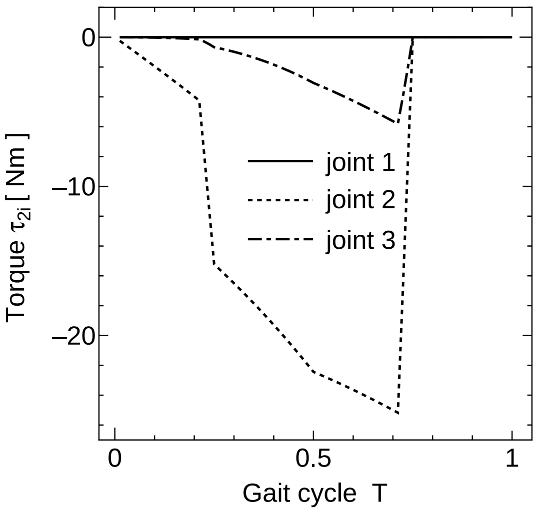

Fig. 7 The joint torques of the leg 2 for case I

4.2 Simulation result for case II

For the case II, $k_{yz} = -0.04$, $\mu^* = 0.03$, $\delta(t)$ is a linear scalar function. Similarly, let $\boldsymbol{p} = \boldsymbol{0}$ and $\boldsymbol{Q} = \boldsymbol{I}$. The foot force distribution obtained in a gait cycle is presented in **Fig. 8**. The ratio of tangential to normal forces at the feet in a gait cycle is

$$0.0427 \leq \frac{\sqrt{f_{xi}^2 + f_{yi}^2}}{f_{zi}} \leq 0.05.$$

Similarly, it is easily found that the three aspects considered are entirely realized in the optimal solution in Fig. 8. The joint torques of leg 2 in a gait cycle are shown in **Fig. 9**.

5. Conclusions

An efficient algorithm for optimal constrained solution of the force distribution problem for a quadruped robot, has been described in this paper. This method contains four steps. First, the friction constraints are transformed from the nonlinear inequalities into a set of linear equalities and linear inequalities that satisfy the condition to prevent leg

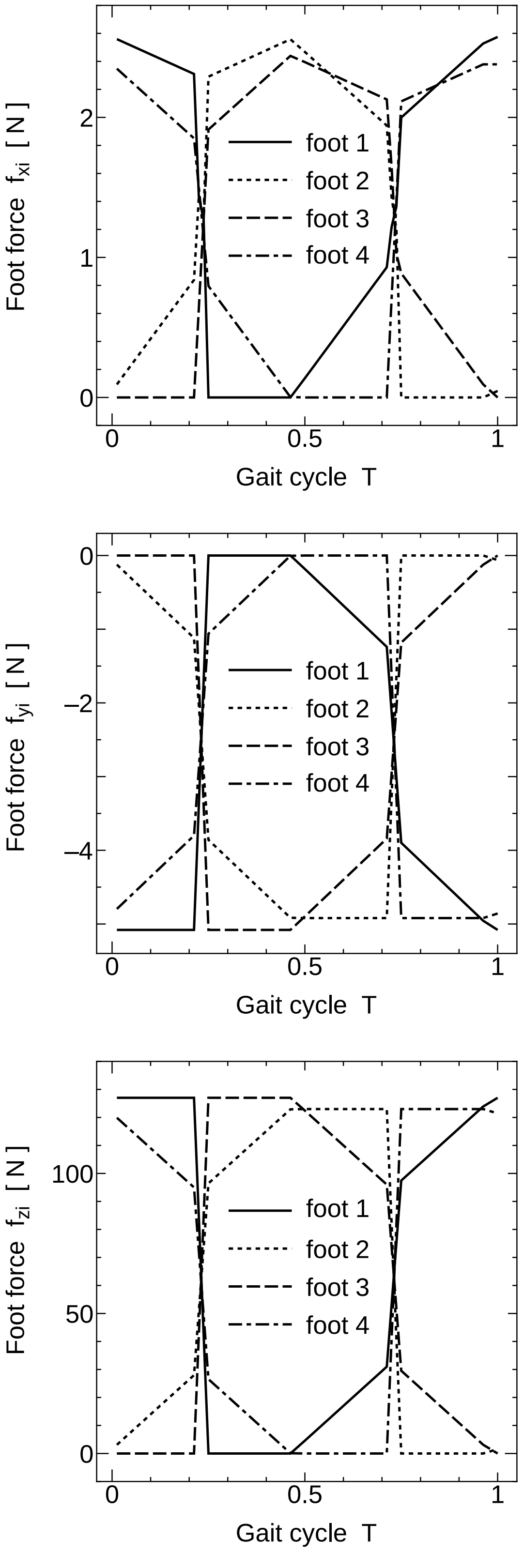

Fig. 8 The foot force distribution for case II

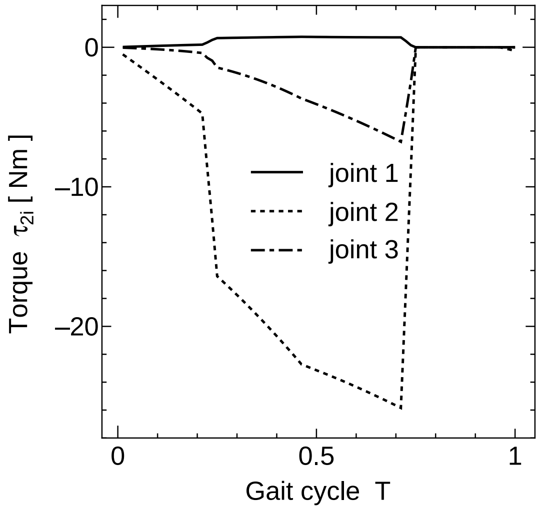

Fig. 9 The joint torques of the leg 2 for case II

slippage. Second, the general solution of the linear equality constraints including those of the friction constraints and the inverse dynamics equations is obtained by transforming the under-specified matrix into row-reduced echelon form. Third, the linear equality constraints of the original problem are eliminated by substituting the general solution into the inequality constraint equations and the objective function. As a consequence, the original problem becomes an optimization of a quadratic function with one unknown quantity. Finally, the optimal constrained solution for the force distribution problem can be obtained by solving the optimization equation. Moreover, in the case of four legs supporting the robot body, the foot forces of the previous swing leg are designed to change smoothly from zero to the ultimate values when the next swing leg is lifted.

Since the problem size of the proposed approach was smaller than those of the problems presented in [8], [14], and [15], it was beyond doubt that the method can be used for the controller in real time. The allowed interval of the design quantity expressed in Eq. (28) resulted from the physical constraints given in Eqs. (3), (4) and (7) or (10), therefore, it was shown that the proposed approach was superior to the Analytical Method [9]–[13] in both the scope of application and quality of solution. The effectiveness of the proposed method was demonstrated through two simulations of the foot force distribution for legs of a quadruped robot.

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